

A short proof of Kneser's addition theorem for abelian groups

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Abstract

Martin Kneser proved the following addition theorem for every abelian group G . If $A, B \subseteq G$ are finite and nonempty, then $|A + B| \geq |A + K| + |B + K| - |K|$ where $K = \{g \in G \mid g + A + B = A + B\}$. Here we give a short proof of this based on a simple intersection union argument.

Throughout we shall assume that G is an additive abelian group. If $A, B \subseteq G$ and $g \in G$, then $A + B = \{a + b \mid a \in A \text{ and } b \in B\}$ and $A + g = g + A = \{a + g \mid a \in A\}$. We define the *stabilizer* of A to be $\mathcal{S}(A) = \{g \in G \mid A + g = A\}$. Note that $\mathcal{S}(A) \leq G$. The goal of this paper is to provide a short proof of the following theorem.

Theorem 1 (Kneser [1]) *If $A, B \subseteq G$ are finite and nonempty and $K = \mathcal{S}(A + B)$, then*

$$|A + B| \geq |A + K| + |B + K| - |K|.$$

Proof. We proceed by induction on $|A + B| + |A|$. Suppose that $K \neq \{0\}$ and let $\phi : G \rightarrow G/K$ be the canonical homomorphism. Then $\mathcal{S}(\phi(A + B))$ is trivial, so by applying induction to $\phi(A), \phi(B)$ we have

$$|A + B| = |K|(|\phi(A) + \phi(B)|) \geq |K|(|\phi(A)| + |\phi(B)| - 1) = |A + K| + |B + K| - |K|.$$

Thus, we may assume $K = \{0\}$. If $|A| = 1$, then the result is trivial, so we may assume $|A| > 1$ and choose distinct $a, a' \in A$. Since $a' - a \notin \mathcal{S}(B) \subseteq \mathcal{S}(A + B) = \{0\}$, we may choose $b \in B$ so that $b + a' - a \notin B$. Now by replacing B by $B - b + a$ we may assume $\emptyset \neq A \cap B \neq A$.

Let $C \subseteq A + B$ and let $H = \mathcal{S}(C)$. We call C a *convergent* if

$$|C| + |H| \geq |A \cap B| + |(A \cup B) + H|.$$

Set $C_0 = (A \cap B) + (A \cup B)$ and observe that $C_0 \subseteq A + B$. Since $0 < |A \cap B| < |A|$, we may apply induction to $A \cap B$ and $A \cup B$ to conclude that C_0 is a convergent. Thus a convergent exists, and we may now choose a convergent C with $H = \mathcal{S}(C)$ minimal. If $H = \{0\}$ then $|A + B| \geq |C| \geq |A \cap B| + |A \cup B| - |\{0\}| = |A| + |B| - 1$ and we are finished. So, we may assume $H \neq \{0\}$ (and proceed toward a contradiction). Since $\mathcal{S}(A + B) = \{0\}$ and $\mathcal{S}(C) = H$, we may choose $a \in A$ and $b \in B$ so that $a + b + H \not\subseteq A + B$. Let $A_1 = A \cap (a + H)$, $A_2 = A \cap (b + H)$, $B_1 = B \cap (b + H)$, and $B_2 = B \cap (a + H)$ and note that $A_1, B_1 \neq \emptyset$. For $i = 1, 2$ let $C_i = C \cup (A_i + B_i)$ and let $H_i = \mathcal{S}(A_i + B_i)$. Observe that if $A_i, B_i \neq \emptyset$, then $H_i = \mathcal{S}(C_i) < H$. The following equation holds for $i = 1$, and it also holds for $i = 2$ if $A_2, B_2 \neq \emptyset$. It follows from the fact that C_i is not a convergent (by the minimality of H), and induction applied to A_i, B_i .

$$\begin{aligned} |(A \cup B) + H| - |(A \cup B) + H_i| &< (|C| + |H| - |A \cap B|) - (|C_i| + |H_i| - |A \cap B|) \\ &= |H| - |A_i + B_i| - |H_i| \\ &\leq |H| - |A_i + H_i| - |B_i + H_i| \end{aligned} \tag{1}$$

If $B_2 = \emptyset$, then $|(A \cup B) + H| - |(A \cup B) + H_1| \geq |(a + H) \setminus (A_1 + H_1)| = |H| - |A_1 + H_1|$ contradicts equation 1 for $i = 1$. We get a similar contradiction under the assumption that $A_2 = \emptyset$. Thus $A_2, B_2 \neq \emptyset$ and equation 1 holds for $i = 1, 2$. If $a + H = b + H$, then $A_1 = A_2$ and $B_1 = B_2$ and we have $|(A \cup B) + H| - |(A \cup B) + H_1| \geq |(a + H) \setminus ((A_1 \cup B_1) + H_1)| \geq |H| - |A_1 + H_1| - |B_1 + H_1|$ which contradicts equation 1. Therefore, $a + H \neq b + H$. Our next inequality follows from the observation that the left hand side of equation 1 is nonnegative, and all terms on the right hand side are multiples of $|H_i|$.

$$|H| \geq |A_i| + |B_i| + |H_i| \tag{2}$$

Let $S = (a + H) \setminus (A_1 \cup B_2)$ and $T = (b + H) \setminus (A_2 \cup B_1)$, and note that S and T are disjoint. The next equation follows from the fact that $A + B$ is not a convergent (by the minimality of H), and induction applied to A_i, B_i .

$$\begin{aligned} |H| &\geq |(A \cup B) + H| + |A \cap B| - |C| \\ &\geq |S| + |T| + |A \cup B| + |A \cap B| - |A + B| + |A_i + B_i| \\ &> |S| + |T| + |A_i| + |B_i| - |H_i| \end{aligned} \tag{3}$$

Summing the four inequalities obtained by taking equations 2 and 3 for $i = 1, 2$ and then dividing by two yields $2|H| > |A_1| + |B_2| + |S| + |A_2| + |B_1| + |T|$. However, $a + H = S \cup A_1 \cup B_2$ and $b + H = T \cup A_2 \cup B_1$. This final contradiction completes the proof. \square

References

- [1] M. Kneser, *Abschätzungen der asymptotischen Dichte von Summenmengen*, Math. Z **58** (1953), 459-484.
- [2] M.B. Nathanson, *Additive Number Theory: Inverse Problems and the Geometry of Sumsets*, GTM 165, Springer, 1996.
- [3] T. Tao, V. Vu, *Additive Combinatorics*, Cambridge University Press, 2006.